UV/IR mixing in Non-Fermi Liquids

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Landau Fermi-Liquid Theory

[Landau (1951)]: A finite density of interacting fermions doesn't depend on specific microscopic dynamics of individual systems:

- **Ground state**: characterized by a sharp Fermi surface (FS) in momentum space
- **Low energy excitations**: weakly interacting quasiparticles around FS

\[
\begin{align*}
(\omega = 0, \quad k_\perp \equiv k - k_F = 0) \\
G_R(\omega, \vec{k}) &= \frac{Z}{\omega - v_F k_\perp + i \Gamma}
\end{align*}
\]

1. Quasiparticle lifetime diverges close to FS $\rightarrow$ Decay rate $\Gamma \sim \omega^2$
2. Electron has a finite overlap with quasiparticle adiabatically connected to non-interacting Fermi gas $\rightarrow$ quasi-particle weight $Z > 0$
Breakdown of FL Theory

- Can be diagonalized in single-particle basis of quasiparticles
- Low energy QFT
- No such basis genuinely interacting QFT

- Fermi Liquid Metals
- Non-Fermi Liquid States

- Heavy fermion compounds near magnetic QCPs, QCP for Mott Transitions, nematic QCP
  - NFL phase at QPT

- Gapless boson by fine-tuning microscopic parameters

- Can arise when FS coupled with a gapless boson

- Bose metals & $\nu = 1/2$ FQH state support fractionalized fermionic excitations + emergent gauge field
  - NFL phases in extended region in parameter space
Unusual Scaling Phenomenology

1. Calculational framework that replaces FL theory needed.

2. QFT of metals $\rightarrow$ low symmetry $+$ extensive gapless modes need to be kept in low energy theories $\rightarrow$ less well understood compared to relativistic QFTs.

\[ \rho(T) - \rho_0 \propto T^\epsilon \]
$\epsilon = 1$ for NFL (yellow)
$\epsilon = 2$ for FL (blue)

[Custers et al, Nature (2003)]
Goals

- Construct minimal field theories that capture universal low-energy physics.
- Understand the dynamics in controlled ways.
- Eventually come up with a systematic classification for NFL's.
- Broadly we have 2 cases:

  - Critical boson mom $q = 0$
    - Ising-nematic QCP, gauge field + spinon FS
  - Critical boson mom $q \neq 0$
    - SDW or CDW critical pts

- Dynamics depends on FS dim ($m$) in addition to spacetime dim ($d+1$). Here we focus on $m$ & $d-m$ dependence for case 1.
Ising-Nematic QPT

From theoretical viewpoint, Ising-nematic (ISN) QCP one of the simplest phase transitions in metals providing a remarkable strongly-coupled NFL with critical fluctuations of ISN order.

- (2+1)-d simple choice change from ■ to ▬ symmetry.

- QPT to nematic states with spontaneously broken point group symmetry order parameter is a real scalar boson with strong qtm fluctuations at QCP.

- FS has $Z_2$ sym

- FS has $Z_4$ sym

- Quantum critical
Dimension as a Tuning Parameter

- For $d < \text{upper critical dim } d_c$, theory flows to interacting NFL at low energies.

- For $d > d_c$, expected to be described by FL.

Choice of regularization scheme for systematic RG in relativistic QFT:

- Locality
- Consistent with many symmetries

Our Dimensional Regularization (DR) scheme:

- Advantage $\Rightarrow$ locality maintained
  
  [ Locality broken in DR scheme of Senthil & Shankar (2009) ]

- Disadvantage $\Rightarrow$ some symmetries broken [ global U(1) ]
Two Patch Theory

Low energy limit
- Fermions coupled with boson with mom tangential to FS
  - scatter tangentially

Circular FS (m=1) - fermions in different patches decoupled except antipodal points

Time-Reflection Invariance assumed

Not true for m-dim FS with \( m > 1 \)

\( k_F \) enters as a dimensionful parameter
Significance of $m$ for $d < d_c$

- $d$ controls strength of qtm fluctuations & $m$ controls extensiveness of gapless modes.

- For $d < d_c$, an emergent locality in mom space for $m = 1$, but not for $m > 1$.

- For $m = 1$, observables local in mom space (e.g. Green’s fns) can be extracted from local patches need not refer to global properties of FS.

- (2+1)-d ISN QCP described by a stable NFL state slightly below $d_c = 5/2$.


- For $m > 1$, UV/IR mixing low-energy physics affected by gapless modes on entire FS effects patch theory cannot capture through renormalization of local properties.
Role of “$k_F$”

- We devise DR extending both dim & co-dim $\rightarrow$ FS with $m > 1$ included naturally.

  [ IM and S-S. Lee, arXiv:1407.0033 ]

- We provide a controlled analysis showing how interactions + UV/IR mixing interplay to determine low-energy scalings in NFL's with general $m$.

- For $m > 1$ $\rightarrow$ size of FS ($k_F$) modifies naive scaling based on patch description $\rightarrow$ $k_F$ becomes a ‘naked scale’. 
At a chosen point $K^*$ on FS: $k_{d-m} \perp$ local $S^m$ \quad its magnitude measures deviation from $k_F$.

$L_{(k)} = (k_{d-m+1}, k_{d-m+2}, ..., k_d)$ \quad tangential along the local $S^m$. 

Patch of m-dim FS of arbitrary shape
Fermions on Antipodal Points

\[ \Psi_j(k) = \begin{pmatrix} \psi_{+,j}(k) \\ \psi_{-,j}^\dagger(-k) \end{pmatrix} \]

right (left) moving fermion with flavour \( j=1,2,...,N \)
Action

2 halves of m-dim FS coupled with one critical boson in (m+1)-space & one time dim:

\[
S = \sum_{s=\pm} \sum_{j=1}^N \int \frac{d^{m+2}k}{(2\pi)^{m+2}} \psi_{s,j}^\dagger(k) \left[ ik_0 + sk_{d-m} + \vec{L}^2_{(k)} + O(\vec{L}^3_{(k)}) \right] \psi_{s,j}(k)
\]

\[
+ \frac{1}{2} \int \frac{d^{m+2}k}{(2\pi)^{m+2}} \left[ k_0^2 + k_{d-m}^2 + \vec{L}^2_{(k)} \right] \phi(-k) \phi(k)
\]

\[
+ \frac{e}{\sqrt{N}} \sum_{s=\pm} \sum_{j=1}^N \int \frac{d^{m+2}k \, d^{m+2}q}{(2\pi)^{2m+4}} \phi(q) \psi_{s,j}^\dagger(k + q) \psi_{s,j}(k)
\]
FS in Terms of Dirac Fermions

Interpret $|L_{(k)}|$ as a continuous flavour

- Each $(m+2)$-d spinor can be viewed as a $(1+1)$-d Dirac fermion

$$
\bar{\Psi}_j(k) = \begin{pmatrix}
\psi_{+,j}(k) \\
\psi_{-j}^{-}(k)
\end{pmatrix}
$$

$$
S = \sum_{j=1}^{N} \int \frac{d^{m+2}k}{(2\pi)^{m+2}} \bar{\Psi}_j(k) \left[ ik_0 \gamma_0 + i \left( k_{d-m} + \bar{L}_{(k)}^2 \right) \gamma_1 \right] \Psi_j(k) \exp \left( \frac{\bar{L}_{(k)}^2}{k_F} \right)
$$

$$
+ \frac{1}{2} \int \frac{d^{m+2}k}{(2\pi)^{m+2}} \left[ k_0^2 + k_{d-m}^2 + \bar{L}_{(k)}^2 \right] \phi(-k) \phi(k)
$$

$$
+ \frac{ie}{\sqrt{N}} \sum_{j=1}^{N} \int \frac{d^{m+2}k d^{m+2}q}{(2\pi)^{2m+4}} \phi(q) \bar{\Psi}_j(k+q) \gamma_1 \Psi_j(k)
$$

mom cut-off
Momentum Regularization along FS

- Compact FS approx by 2 sheets of non-compact FS with a momentum regularization suppressing modes far away from $\pm K^*$:

We keep dispersion parabolic but exp factor effectively makes FS size finite by damping $|\tilde{L}_{(k)}| > k_F^{1/2}$ fermion modes.
Theory in General Dimensions

Add (d-m-1) spatial dim

**co-dimensions**

\[ k_0 \rightarrow \vec{K} \equiv (k_0, k_1, \ldots, k_{d-m-1}) \]

\[ \gamma_0 \rightarrow \vec{\Gamma} \equiv (\gamma_0, \gamma_1, \ldots, \gamma_{d-m-1}) \]

\[ \gamma_1 (k_{d-m} + \vec{L}_{(k)}^2) \rightarrow \gamma_{d-m} \delta_k \]

\[ \delta_k = k_{d-m} + \vec{L}_{(k)}^2 \]

\[
S = \sum_j \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \Psi_j(k) \left[ i\vec{\Gamma} \cdot \vec{K} + i\gamma_{d-m} \delta_k \right] \Psi_j(k)
\]

\[
+ \frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left[ |\vec{K}|^2 + k_{d-m}^2 + \vec{L}_{(k)}^2 \right] \phi(-k)\phi(k)
\]

\[
+ \frac{ie}{\sqrt{N}} \sum_j \int \frac{d^{d+1}k \, d^{d+1}q}{(2\pi)^{2d+2}} \phi(q) \Psi_j(k + q) \gamma_{d-m} \Psi_j(k)
\]
Applying DR

- There is an implicit UV cut-off $\Lambda$ for $k$ with $k << \Lambda << k_F$.

- $k_F$ sets FS size;
  $\Lambda$ sets the largest energy fermions can have $\perp$ FS.

- We consider RG flow by changing $\Lambda$ & requiring low-energy observables independent of it.

- To access perturbative NFL, we fix $m$ & tune $d$ towards a critical dim $d_c$ at which qtm corrections diverge logarithmically in $\Lambda$. 

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![Diagram](image)
Critical Dimension

- Naïve critical dim ➔ scaling dim of \( e = 0 \):
  \[
  d'_c = \frac{4 + m}{2}
  \]

- True critical dim ➔ one-loop fermion self-energy \( \Sigma_1(q) \) blows up logarithmically:
  \[
  d_c = m + \frac{3}{m + 1}
  \]
One-Loop Results for \( d = d_c - \epsilon \)

**Effective coupling & control parameter in loop expansions**

\[
e_{\text{eff}} \equiv \frac{e^{2(m+1)/3}}{(m-1)(2-m)\tilde{k}_F^6} k_F = \mu \tilde{k}_F
\]

**Fixed points**

\[
\tilde{\beta} \equiv \frac{\partial e_{\text{eff}}}{\partial \ln \mu} = \frac{(m+1)(u_1 e_{\text{eff}} - N\epsilon)}{3N - (m+1)u_1 e_{\text{eff}}} = 0
\]

**Interacting Fixed Point**

\[
e_{\text{eff}}^* = \frac{N\epsilon}{u_1}
\]

\[
z^* = 1 + \frac{(m+1)\epsilon}{3}
\]

\[
\eta_\psi^* = \eta_\phi^* = -\frac{\epsilon}{2}
\]

**Dynamical critical exponent**

**Anomalous dimensions for fermions & boson**
Stable NFL Fixed Point

Small $e_{\text{eff}}$ expansion:

\[ \tilde{\beta} = -\frac{(m + 1)\epsilon}{3} e_{\text{eff}} + \frac{(m + 1)\{3 - (m + 1)\epsilon\} u_1}{9N} e_{\text{eff}}^2 + O(e_{\text{eff}}^3) \]

Low energy limit
- theory flows to a Stable NFL Fixed Point
- $e_{\text{eff}}$ marginal at $d_c$

For small $\epsilon$, interacting f.p. perturbatively accessible though $\epsilon$ has +ve scaling dim for $1 < m < 2$
Two-Loop Results: Boson Self-Energy

For $m > 1$

\[ \Pi_2(q) \sim \frac{e^2}{6N} \frac{k_F^{m-1}}{|\vec{L}(q)|^2} \pi^2 \frac{e_{eff}^m}{k_F^{2(m+1)}} \]

$\bullet$ $k_F$ suppressed $\bullet$ no correction at 2-loop

For $m = 1$ $\bullet$ UV-finite, gives a finite correction $\bullet$ \[ \Pi_2(q) \sim \left( \frac{e^2}{N |L(q)|} \right) e_{eff} \]
Two-Loop Results: Fermion Self-Energy

- For $m > 1$: $\Sigma_2(q) \sim k_F - \text{suppressed}$
- No correction at 2-loop
- For $m = 1$: UV-divergent
Pairing Instabilities of Critical FS States

- Regular FL unstable to arbitrary weak interaction in BCS channel leading to Cooper pairing ➔ How about a critical FS?

- Metlitski, Mross, Sachdev & Senthil [arXiv:1403.3694] studied SC instability in (2+1)-d for NFL.

- Chung, IM, Raghu & Chakravarty [Phys. Rev. B 88, 045127 (2013)] found Hatree-Fock soln of self-consistent gap eqn for a FS coupled to a transverse U(1) gauge field in (3+1)-d.

- We want to consider ISN scenario for \( m \geq 1 \).

[IM and S-S. Lee, in progress]
Superconducting Instability

Add generic 4-fermion terms to analyse SC instability:

\[
S_{4f} = \mu_d \sum_{j,j'} \int \frac{d^{d+1}k \ d^{d+1}k_1 \ d^{d+1}k_2}{(2\pi)^{3d+3}}
\left[
V_1 \left\{ \bar{\Psi}_j(k_1 + k) \gamma_{d-m} \Psi_j(k_1) \right\} \left\{ \bar{\Psi}_{j'}(k_2 - k) \gamma_{d-m} \Psi_{j'}(k_2) \right\}
\right]
\]

\[
+ \ V_2 \sum_{\mu=0}^{d-m-1} \left\{ \bar{\Psi}_j(k_1 + k) \Gamma_{\mu} \Psi_j(k_1) \right\} \left\{ \bar{\Psi}_{j'}(k_2 - k) \Gamma_{\mu} \Psi_{j'}(k_2) \right\}
\]

\[
+ \ V_3 \sum_t \left\{ \bar{\Psi}_j(k_1 + k) \sigma_t \Psi_j(k_1) \right\} \left\{ \bar{\Psi}_{j'}(k_2 - k) \sigma_t \Psi_{j'}(k_2) \right\}
\]

\[
(\sigma_t, \Gamma_{\mu}, \gamma_{d-m}) \in \{\mathbb{I}_{2 \times 2}, \sigma_x, \sigma_y, \sigma_z\}
\]
Beta-Fns for $V_a$'s

- Scatterings in pairing channel enhanced by volume of FS $\sim (k_F)^{m/2}$.

- Effective coupling that dictates potential instability:
  $$\tilde{V}_a = \tilde{k}_F^{m/2} V_a$$

- $\tilde{V}_a$ marginal at co-dim $d - m = 1$.

- For $d-m>1$ → no perturbative instability for sufficiently small $\epsilon = d_c - d$.

- When $d - m - 1 \lesssim \epsilon$ & $d - d_c \sim \epsilon$ → interaction plays an imp role to determine pairing instability.
Epilogue

- RG analysis for QFTs with FS scaling behaviour of NFL states in a controlled approx.

- m-dim FS with its co-dim extended to a generic value stable NFL fixed points identified using $\epsilon = d_c - d$ as perturbative parameter.

- SC instability in such systems as a fn of dim & co-dim of FS.

- Key point $k_F$ enters as a dimensionful parameter unlike in relativistic QFT modify naive scaling arguments.

- Effective coupling constants combinations of original coupling constants & $k_F$. 
Thank you for your attention!
A Physical Realization for $d=3, m=1$

Turn on p-wave SC order parameter
- gap out the cylindrical FS except for a line node
(a) m-dim FS embedded in d-dim mom space.

(b) Spinor has 2 bands: 

\[ E_k = E_F \pm \sqrt{\sum_{i=1}^{(d-m-1)} k_i^2 + \delta_k^2} \]

For each \( L_{(k)} \) Dirac point \( \equiv (k_1 = 0, k_2 = 0, ..., k_{d-m} = - (L_{(k)})^2) \) around which energy disperses linearly like a Dirac fermion in the (d-m)-dim subspace.
Two-point Fns at IR Fixed Point

Using RG eqns

\[
\langle \phi(-k)\phi(k) \rangle = \frac{1}{(\tilde{L}^2_{(k)})^{2\Delta\phi}} f_D \left( \frac{|\tilde{K}|^{1/z^*}}{\tilde{L}^2_{(k)}}, \frac{k_{d-m}}{k_F}, \frac{\tilde{L}^2_{(k)}}{k_F} \right)
\]

\[
\langle \psi(k)\overline{\psi}(k) \rangle = \frac{1}{|\delta_k|^{2\Delta\psi}} f_G \left( \frac{|\tilde{K}|^{1/z^*}}{\delta_k}, \frac{\delta_k}{k_F}, \frac{\tilde{L}^2_{(k)}}{k_F} \right)
\]

One-loop order

\[
f_D(x, y, z) = \left[ 1 + \beta_d \tilde{e}^{\frac{3}{m+1}} x^{\frac{3}{m+1}} z^{-\frac{3(m-1)}{2(m+1)}} \right]^{-1}
\]

\[
f_G(x, y, z) = -i \left[ C (\hat{\Gamma} \cdot \hat{K}) x + \gamma_{d-m} \right]^{-1}
\]