Resurgence, Trans-series and Quantum Field Theory

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Aspen 2015 Winter Conference: Progress and Application of Modern QFT

GD & M. Ünsal, 1210.2423, 1210.3646, 1306.4405, 1401.5202

GD, lectures at CERN 2014 Winter School

also with: G. Başar, A. Cherman, D. Dorigoni, R. Dabrowski: 1306.0921, 1308.0127, 1308.1108, 1405.0302, 1501.05671

Physical Motivation

- infrared renormalon puzzle in asymptotically free QFT
  (i) IR renormalons $\Rightarrow$ perturbation theory ill-defined
  (ii) $\mathcal{I}\bar{\mathcal{I}}$ interactions $\Rightarrow$ instanton-gas ill-defined
- non-perturbative physics without instantons: physical meaning of non-BPS saddle configurations
- "exact" asymptotics in QFT, localization & string theory

Bigger Picture

- non-perturbative definition of nontrivial QFT in continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view semiclassical expansions as potentially exact

Resurgent Trans-Series

- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

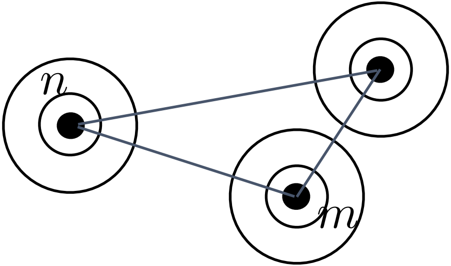
- J. Écalle (1980): set of functions closed under:

(Borel transform) + (analytic continuation) + (Laplace transform)

- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponentially improved asymptotics

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



recap: rough basics of Borel summation

(i) divergent, alternating:

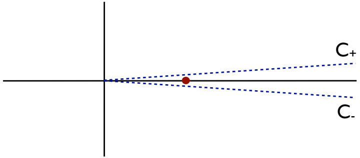
$$\sum_{n=0}^{\infty}(-1)^n\, n!\, g^{2n} = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+g^2\, t}$$

(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n!\, g^{2n} = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1-g^2\, t}$$

$\Rightarrow$ ambiguous imaginary non-pert. term: $\pm\frac{i\,\pi}{g^2}\, e^{-1/g^2}$

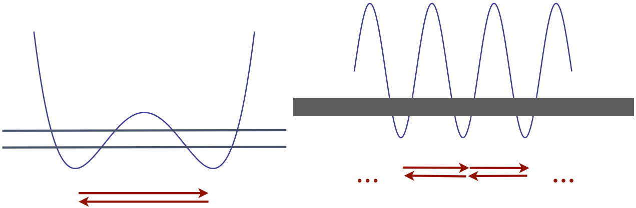
avoid singularities on $\mathbb{R}^+$: *lateral* Borel sums:



$\theta = 0^{\pm} \longrightarrow$ non-perturbative ambiguity: $\pm\mathrm{Im}[\mathcal{B}f(g^2)]$

challenge: use physical input to resolve ambiguity

QM Analogue of IR Renormalon Problem in QFT



- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

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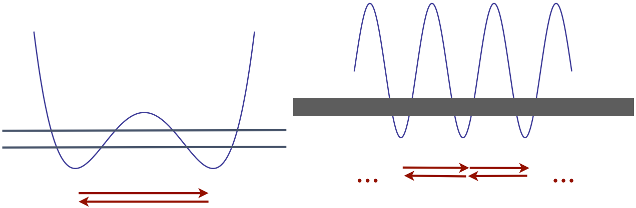


- degenerate vacua: double-well, Sine-Gordon, ...

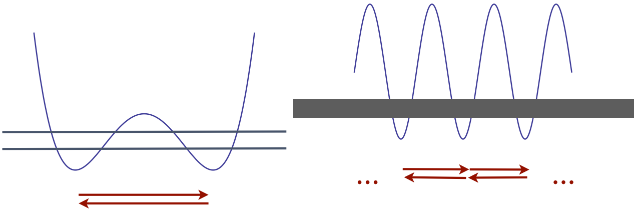
splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel-summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

“Bogomolny/Zinn-Justin mechanism”

Bogomolny 1980; Zinn-Justin 1980



- degenerate vacua: double-well, Sine-Gordon, ...

  1. perturbation theory non-Borel-summable:
     ill-defined/incomplete

  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

- regularize *both* by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

“resurgence” $\Rightarrow$ cancellation to all orders

Towards Resurgence in QFT: Renormalons

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

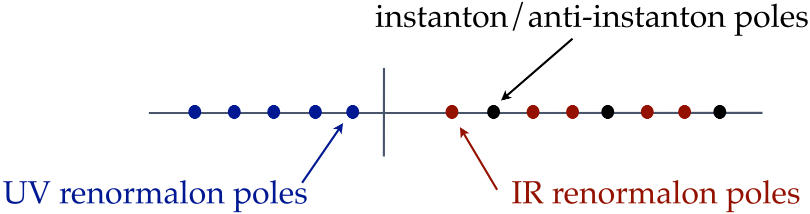
QFT: new physical effects occur, due to running of couplings with momentum

- asymptotically free QFT
- faster source of divergence: “renormalons”
- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow$ $\pm i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow$ $\pm i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

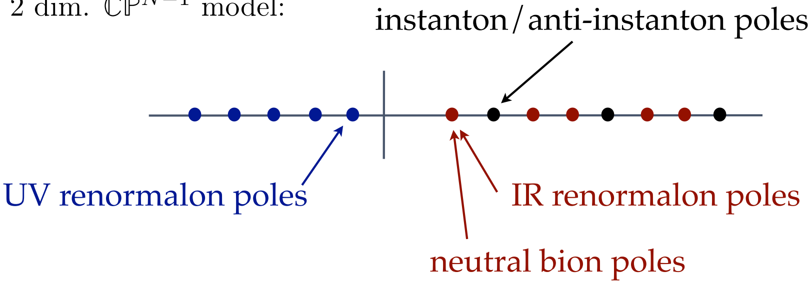
’t Hooft, 1980; David, 1981

IR Renormalon Puzzle in Asymptotically Free QFT

(Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

resolution: there is another problem with the non-perturbative instanton gas analysis: scale modulus of instantons

- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

semiclassical realization of IR renormalons

- non-perturbative sector: bion-bion amplitudes

$$\left[\mathcal{I}_i\bar{\mathcal{I}}_i\right]_\pm = \left(\ln\left(\frac{g^2\,N}{8\pi}\right)-\gamma\right)\frac{16}{g^2\,N}\,e^{-\frac{8\pi}{g^2\,N}} \pm i\pi\,\frac{16}{g^2\,N}\,e^{-\frac{8\pi}{g^2\,N}}$$

- perturbative sector: lateral Borel summation

$$B_\pm\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\,B\mathcal{E}(t)\,e^{-t/g^2} = \mathrm{Re}\,B\mathcal{E}(g^2) \mp i\pi\,\frac{16}{g^2\,N}\,e^{-\frac{8\pi}{g^2\,N}}$$

exact cancellation !

explicit application of resurgence to nontrivial QFT

The Bigger Picture

question: origin of resurgent behavior in QM and QFT ?

- uniform WKB (GD, Ünsal, 1306.4405, 1401.5202) $\Rightarrow$

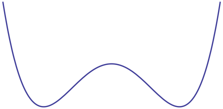
(i) trans-series structure is generic
(ii) all multi-instanton effects encoded in perturbation theory

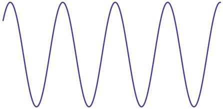
- basic property of all-orders steepest descents integrals

- Lefschetz thimbles: analytic continuation of path integrals

Uniform WKB & Resurgent Trans-series (GD/MÜ:1306.4405, 1401.5202)

$$-\frac{d^2}{dx^2}\psi + \frac{V(g\,x)}{g^2}\psi = E\,\psi \to -g^4\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2\,E\,\psi(y)$$





- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

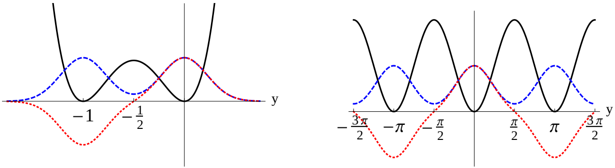
- ansatz (with parameter ν): $\psi(y) = \frac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$

Uniform WKB & Resurgent Trans-Series

- perturbative expansion for E and $u(y)$:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

- $\nu = N$: usual perturbation theory (not Borel summable)
- global analysis $\Rightarrow$ boundary conditions:



- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow \quad g^2 \to e^{\pm i\,\epsilon}\, g^2$
- trans-series encodes analytic properties of D_ν

  $\Rightarrow$ generic and universal

Connecting Perturbative and Non-Perturbative Sector

Zinn-Justin/Jentschura: generate *entire trans-series* from

(i) perturbative expansion $E = E(N, g^2)$
(ii) single-instanton fluctuation function $\mathcal{P}(N, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

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in fact ... (GD, Ünsal, 1306.4405, 1401.5202)

$$\mathcal{P}(N, g^2) = \exp\left[S \int_0^{g^2} \frac{dg^2}{g^4} \left(\frac{\partial E(N, g^2)}{\partial N} - 1 + \frac{\left(N + \frac{1}{2}\right) g^2}{S} \right)\right]$$

$\Rightarrow$ perturbation theory $E(N, g^2)$ encodes everything !

Resurgence at work

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle determined by those about the vacuum saddle, to all fluctuation orders

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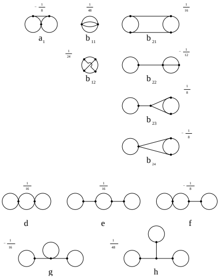


- fluctuation about $\mathcal{I}$ for double-well:

2-loop (Shuryak/Wöhler, 1994); 3-loop

(Escobar-Ruiz/Shuryak/Turbiner, arXiv:1501.03993)

$$E/S/T: \quad e^{-\frac{S_0}{g}}\left[1 - \frac{71}{72}\,g - 0.607535\,g^2 - \dots\right]$$

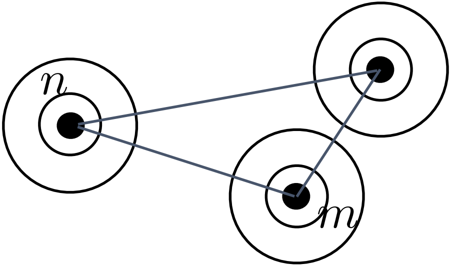
$$\text{D/Ü}: \quad e^{-\frac{S_0}{g}}\left[1 + \frac{1}{72}\,g\left(-102N^2 - 174N - 71\right)\right.$$

$$\left. + \frac{1}{10368}g^2\left(10404N^4 + 17496N^3 - 2112N^2 - 14172N - 6299\right) + \dots\right]$$

- known for all N and to essentially any loop order !

Connecting Perturbative and Non-Perturbative Sector

all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



why ? turn to path integrals ... (also: QFT)

All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

  <u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- exact resurgent relation between fluctuations about $n^{\rm th}$ saddle and about neighboring saddles m

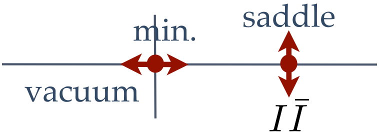
$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz\, e^{-\frac{1}{g^2}\sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$

$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \dots\right.$$

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- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}g^2 + \frac{9}{32}g^4 - \frac{75}{128}g^6 + \dots\right)$$

- fluctuations about the two saddles are explicitly related
- could something like this work for path integrals ?

Resurgence in Path Integrals (GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\text{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

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- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n!\left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{6g^2}}\left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

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remaining path integral has real measure:

(i) Monte Carlo

(ii) semiclassical expansion

(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ... gradient flow

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

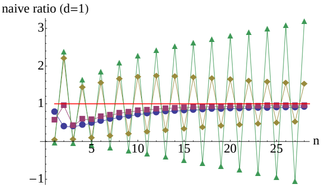
Ghost Instantons: Analytic Continuation of Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

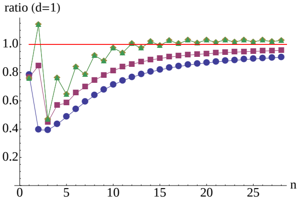
$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x \, e^{-S[x]} = \int \mathcal{D}x \, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)\right)}$$

- doubly periodic potential: *real* & *complex* instantons

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons



with ghost instantons

- complex instantons directly affect perturbation theory, even though they are not in the original path integral measure

Non-perturbative Physics Without Instantons

Dabrowski, GD, 1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127, 1403.1277

- $O(N)$ & principal chiral model have no instantons !
- Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, principal chiral model, ... all have non-BPS solutions with finite action

(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

- "unstable": negative modes of fluctuation operator
- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Resurgence and Localization

(Drukker et al, 1007.3837; Mariño, 1104.0783; Aniceto, Russo, Schiappa, 1410.5834)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models $\Rightarrow$ resurgent asymptotics

- 3d Chern-Simons on $\mathbb{S}^3 \to$ matrix model

$$Z_{CS}(N,g) = \frac{1}{\mathrm{vol}(U(N))} \int dM \exp\left[-\frac{1}{g}\,\mathrm{tr}\left(\frac{1}{2}\left(\ln M\right)^2\right)\right]$$

- ABJM: $\mathcal{N}=6$ SUSY CS, $G = U(N)_k \times U(N)_{-k}$

$$Z_{ABJM}(N,k) = \sum_{\sigma \in S_N} \frac{(-1)^{\epsilon(\sigma)}}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi k} \frac{1}{\prod_{i=1}^{N} 2\mathrm{ch}\left(\frac{x_i}{2}\right)\,\mathrm{ch}\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)}$$

- $\mathcal{N}=4$ SUSY Yang-Mills on $\mathbb{S}^4$

$$Z_{SYM}(N,g^2) = \frac{1}{\mathrm{vol}(U(N))} \int dM \exp\left[-\frac{1}{g^2}\,\mathrm{tr} M^2\right]$$

Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- $a = \langle \text{scalar} \rangle \;, \quad a_D = \langle \text{dual scalar} \rangle \;, \quad a_D = \frac{\partial \mathcal{F}}{\partial a}$
- Nekrasov prepotential:

$$\mathcal{F}_{NS}(a,\hbar) \;=\; \mathcal{F}^{class.}(a,\hbar) + \mathcal{F}^{pert.}(a,\hbar) + \mathcal{F}^{inst.}(a,\hbar)$$

$$\mathcal{F}^{inst} \sim \frac{\hbar^2}{2\pi i}\left(\frac{\Lambda^4}{16a^4} + \frac{21\Lambda^8}{256a^8} + \dots\right) + \frac{\hbar^4}{2\pi i}\left(\frac{\Lambda^4}{64a^6} + \frac{219\Lambda^8}{2048a^{10}} + \dots\right) + \dots$$

$$\mathcal{F}^{class} + \mathcal{F}^{pert} \sim -\frac{a^2}{2\pi i}\log\frac{a^2}{\Lambda^2} - \frac{\hbar^2}{48\pi i}\log\frac{a^2}{2\Lambda^2} + \hbar^2\sum_{n=1}^{\infty} d_{2n}\left(\frac{\hbar}{a}\right)^{2n}$$

- encoded in the Mathieu equation:

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2\cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- all-orders WKB action: (Dunham, 1932)

$$a=\sum_{n=0}^{\infty}\hbar^{2n}\,a_n(u)$$

- Bohr-Sommerfeld in large a (electric) region:

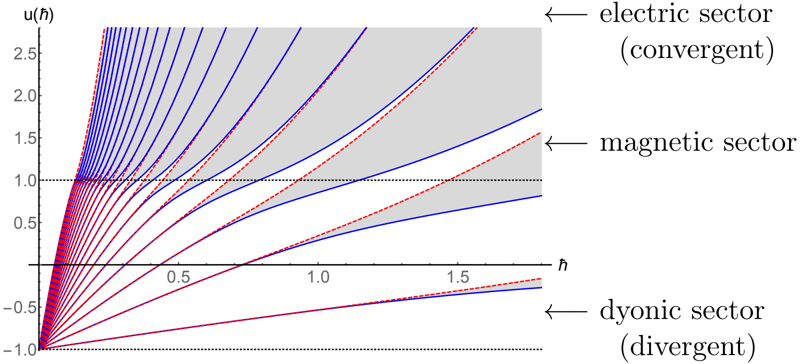
$$\text{invert}\quad a=\frac{N}{2}\,\hbar\quad\Longrightarrow\quad u=u(N,\hbar)=u(a,\hbar)$$

- Matone relation:

$$u(a,\hbar)=\frac{i\pi}{2}\Lambda\frac{\partial\mathcal{F}_{NS}(a,\hbar)}{\partial\Lambda}-\frac{\hbar^2}{48}$$

Resurgence in $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ Theories (Başar, GD, 1501.05671)

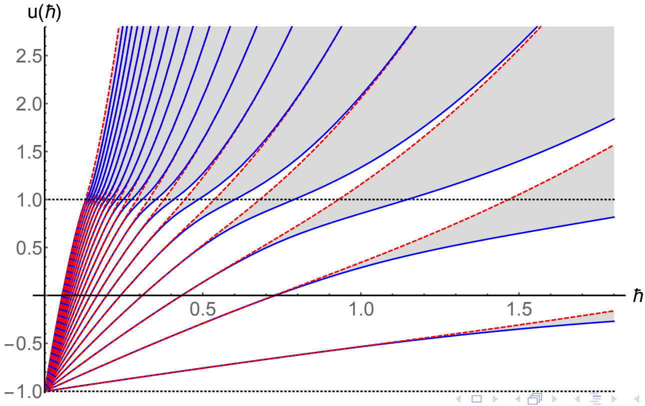
- Mathieu & Lamé eqs encode Nekrasov prepotential



- all-orders WKB: $u = u(N, \hbar) \sim \sum_{n=0}^{\infty} \hbar^{2n} p_{n+1}(N)$
- 't Hooft coupling: $\lambda \equiv N\,\hbar$
- very different physics for $\lambda \gg 1$, $\lambda \sim 1$, $\lambda \ll 1$

Uniform WKB and $\mathcal{N} = 2$ SUSY SU(2) (Başar, GD, 1501.05671)

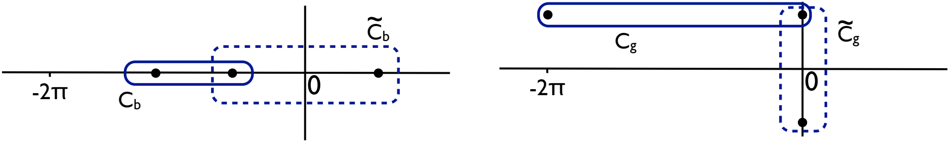
- Bohr-Sommerfeld misses non-perturbative physics
- misses band and gap splittings
- smooth transition through magnetic region ?
- explicitly connect weak and strong coupling



Uniform WKB and $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- universal band/gap splitting: (Landau, Dykhne, Keller, ...)

$$\Delta u(N,\hbar) \sim \frac{2}{\pi}\frac{\partial u}{\partial N}\, \exp\left[-\frac{2\pi}{\hbar}\,\text{Im}\, a_D\right]$$



- dyonic sector: $\Delta u(N,\hbar) \sim \frac{64}{\sqrt{\pi}}\left(\frac{32}{\hbar}\right)^{N-\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right]$
- electric sector: $\Delta u(N,\hbar) \sim \frac{N\hbar^2}{2\pi}\left(\frac{e}{\hbar N}\right)^{2N}$
- magnetic sector: bands & gaps $\sim O(\hbar)$ (equal !)
- resurgent analysis covers all sectors, uniformly

Perturbative/Non-perturbative connection

- Zinn-Justin: $B(u,\hbar)$, $A(u,\hbar)$ determine full trans-series
- GD, Ünsal: $u(B,\hbar)$ encodes $A(B,\hbar)$:

$$\frac{\partial u}{\partial B} = -\frac{\hbar}{16}\left(2B + \hbar\frac{\partial A}{\partial \hbar}\right)$$

- simple proof from Nekrasov $\mathcal{F}$ and Matone relation

$$u \sim \Lambda\frac{\partial \mathcal{F}}{\partial \Lambda} \quad \Rightarrow \quad \frac{\partial u}{\partial a} \sim \Lambda\frac{\partial}{\partial \Lambda}\frac{\partial \mathcal{F}}{\partial a} = \Lambda\frac{\partial a_D}{\partial \Lambda}$$

- identifications:

$$a \leftrightarrow \frac{\hbar}{2}B \quad , \quad a_D \leftrightarrow \frac{\hbar}{4\pi}A + \text{shift} \quad , \quad \Lambda \sim \frac{1}{\hbar}$$

- quantum geometry: $a(u,\hbar)$ and $a_D(u,\hbar)$ related
- **uniform** WKB spans electric/magnetic/dyonic sectors

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis
- trans-series 'encode' all information; expansions about different saddles are intimately related
- there is extra un-tapped 'magic' in perturbation theory
- matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory (Başar, GD, 1501.05671)
- fundamental property of steepest descents
- moral: go complex and consider all saddles, not just minima