Using a Conformal Basis on the Light-Cone To solve a field theory.

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Outline

1. Introduction - Effective Conformal Dominance and and the conformal basis on the light-cone (LC).
2. Use this basis to solve 2D QCD: Fundamental Fermions coupled to a gauge field at strong coupling.
3. Progress in formulating the LC conformal basis in 3D and results for a free massive scalar theory.
4. Some comments on non-Lagrangian theories.
5. Conclusions.

To a CFT add a relevant operator which results in a mass gap in IR.

In a theory with a mass gap, what is the relation between:

UV Primary $\mathcal{O}_\Delta(x)$ ⟷ ? ⟷ $|\psi_{light}; m^2 \sim m^2_{Gap}\rangle$

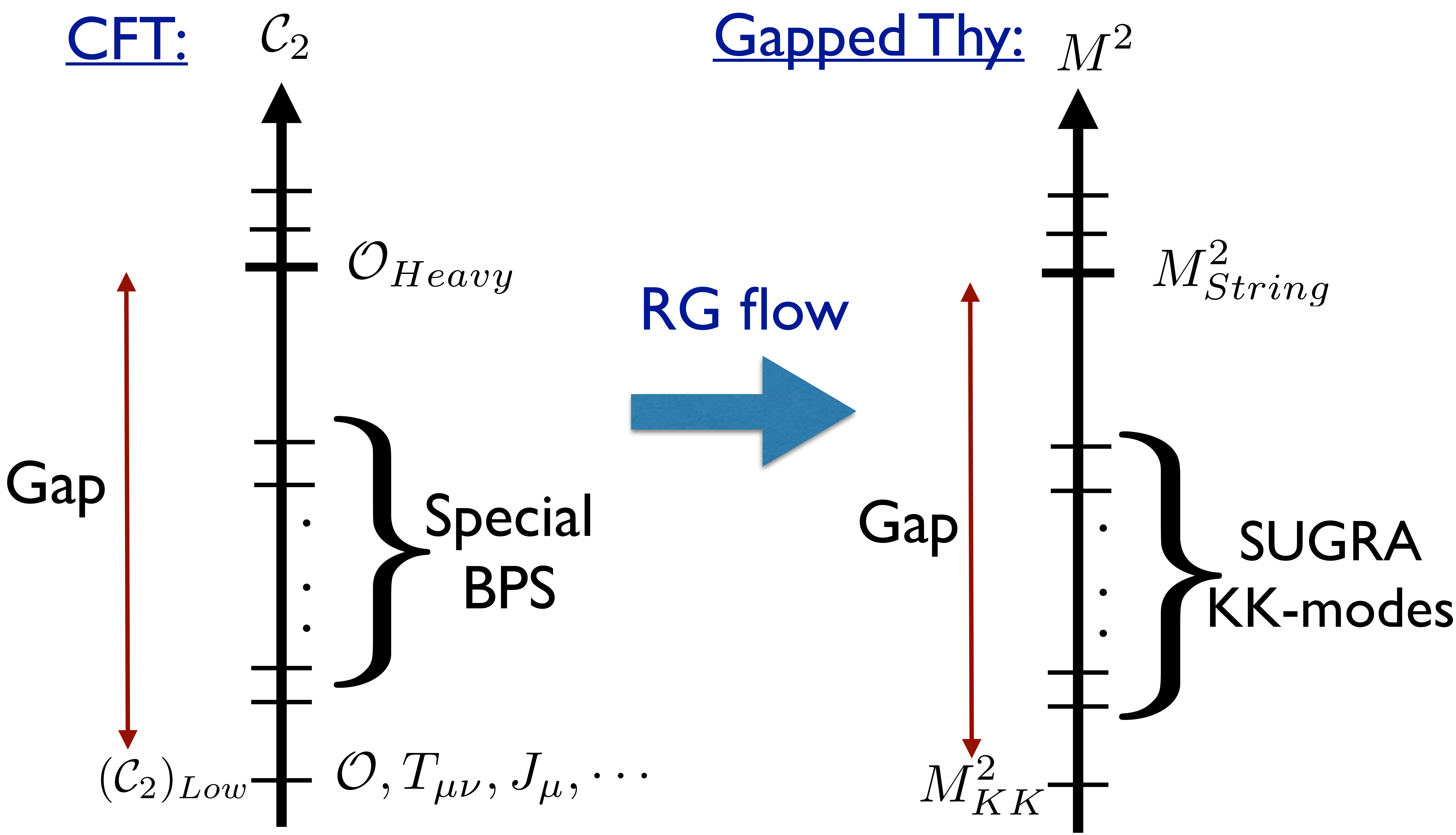
E-state of the Conformal Casimir

E-state of $M^2 \equiv P_\mu P^\mu$

Ex: $\mathcal{C}_2 = \Delta(\Delta - d)$ for a scalar

(AdS Mass)

Known holographic RG flows



AdS/CFT: Light bound-states are mostly created by lowest "conformal harmonics."

This motivates a question

Within a particular sector:

$$\langle \Omega | \mathcal{O}_{\Delta}(0) | \psi_{light} \rangle \equiv f(\Delta)\,(Canonical\ Size)$$

What is the behavior

$f(\Delta) = ?$ as $\Delta \to \infty$?

Naively from holography: $f(\Delta) \sim \frac{1}{\Delta^2}$

(since AdS masses are related to scaling dim in QFT)

Indeed - true for the free 3D massive scalar.

A conjecture: $f(\Delta) \sim e^{-c\Delta}$ w/ Fitzpatrick, Kaplan & Randall

in an interesting class of systems!

True for 2D QCD

Decoupling of high-dimension ops:
"Effective Conformal Dominance"

A QM analogy:

Take a spherically symmetric potential
an break spherical symmetry
with a low spherical harmonic

Ex: turn on E-field in the z-direction: $\Delta V = \epsilon \cos(\theta)$

A low-harmonic mixes low-harmonics
only with other low-harmonics: $\langle \ell + 1 | \Delta V | \ell \rangle \approx \epsilon/2$

Final GS-state has small overlap
with high harmonics: $\langle \ell | \psi_{GS} \rangle \sim \epsilon^{\ell}$

Back to conformal symm:

A relevant operator is always
a low "conformal-harmonic".

OPE convergence:
$C_{LL'H} \ll 1$

→ Light states dominated
by low conf harmonics!

Basic idea: Since higher conformal harmonics decouple rapidly from low-mass states, we can try to use them as a basis to solve the theory:

$$|\psi_{light}\rangle = \sum_i \langle \mathcal{O}_i | \psi_{light}\rangle | \mathcal{O}_i \rangle$$

Truncate to $|\mathcal{O}_i\rangle$ with $\Delta < \Delta_{max}$

Building the basis:

Vacuum after conformal breaking → $\langle \Omega | \mathcal{O}_i | \psi_{light} \rangle$

But the basis is more naturally built on the conformal preserving vacuum.

Our approach: Use light-cone quantization

$$x^{\pm} = x^0 \pm x^1$$

LC-time: $x^+ \to P_+$ LC-Hamiltonian

LC-space: $x^- \to P_-$ LC-momentum

Light-cone: $|\Omega\rangle = |0\rangle_{CFT}$

$$P_+ = P_{+,CFT} + P_{+,Rel-Op} \quad \text{w/} \quad [P_{+,Rel-Op},\ P_-] = 0$$

Now, $P_-|0\rangle_{CFT} = 0$ but $P_- > 0$ for other $|\psi\rangle$ (NEC)

$$\langle\psi|P_{+,Rel-Op}|0\rangle_{CFT} = 0$$

Basis in d-dim (E-states of Conf.-Casimir and momentum):

$$|\mathcal{O}_i(p_-, \vec{p}_\perp)\rangle \sim \int dx^- d^{d-2}x_\perp\ e^{ipx}\mathcal{O}_i(x^+ = 0, x^-, \vec{x}_\perp)|0\rangle$$

Primary: $\langle\mathcal{O}_i(\vec{p})|\mathcal{O}_j(\vec{p'})\rangle = \delta_{ij}\delta^{d-1}(p - p')$

Evidence for effective conformal dominance from 2D QCD at Finite-N.

Why this model?

1. An interesting assymp. free theory w/ strong-coupling in the IR and a set of bound-states.

2. Gluons have no d.o.f. : in Light-cone gauge one can easily integrate them out.

SU(N): $$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F^2) + i\bar{\Psi}\gamma^\mu D_\mu \Psi$$

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left((\partial_- A_+)^2\right) + i\psi^\dagger D_+ \psi + i\chi^\dagger \partial_- \chi$$

$$P_- = 2i \int dx^- \psi^\dagger \partial_- \psi$$

$$P_+ = -g^2 \int dx^- \psi^\dagger T^a \psi \frac{1}{\partial_-^2} \psi^\dagger T^a \psi$$

$$[g] \sim (Mass)$$

3. Interesting: Relativistic bound-states which do not contain a definite number of particles (much like real QCD).

4. Has been solved at large-N by 't Hooft.

A conformal basis for the H-space

Quasi-primary ops of the free fermion theory:

$$[K^{\pm}, \mathcal{O}_{\Delta,s}(x)] = i\left((x^{\pm})^2\partial_{\pm} + (\Delta \pm s)\, x^{\pm}\right)\mathcal{O}_{\Delta,s}(x)$$

$$\mathcal{O}(x^-) = \frac{1}{N^k}\sum_{\sum s_i = n} c_{s_1,s_2,\ldots,s_{2k}}\left(\partial^{s_1}\psi^{\dagger}_{i_1}\partial^{s_2}\psi_{i_1}\right)\ldots\left(\partial^{s_{2k-1}}\psi^{\dagger}_{i_k}\partial^{s_{2k}}\psi_{i_k}\right)$$

Fourier transform x^- :

$$\langle p_1, p_2, \ldots, p_k|\tilde{\mathcal{O}}_{n+k/2}(P)|0\rangle = \delta\left(\sum p_i - P\right)\, f(p_1, p_2, \ldots, p_k)$$

The amplitude $f(p_1, p_2, ..., p_k)$ is a polynomial in the p_i

The K-killing eqn determines this amplitude to be related to Jacobi polynomials.
(which form a basis on the simplex spanned by p_i)

For example, for 2-fermions: $P_k(2x-1), \; x = \frac{p_1}{P}$

Eff. Conf. Dom. → Light-states are made from low-degree polys

Goal - Diagonalize: $M^2_{ij} = \langle \mathcal{O}_i | 2P^+P^- | \mathcal{O}_j \rangle$

Basis:

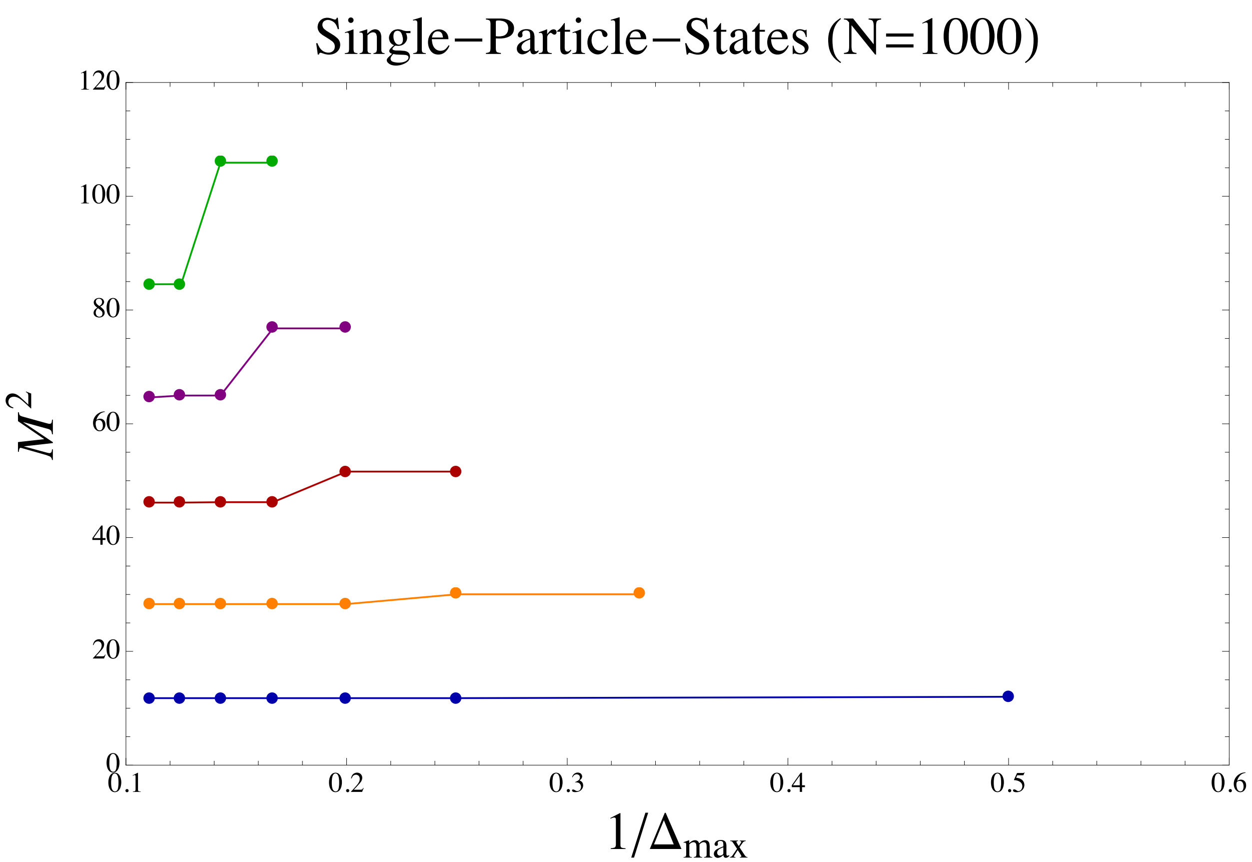
$$\mathcal{O}^{(1)} \sim \psi^\dagger \psi,$$

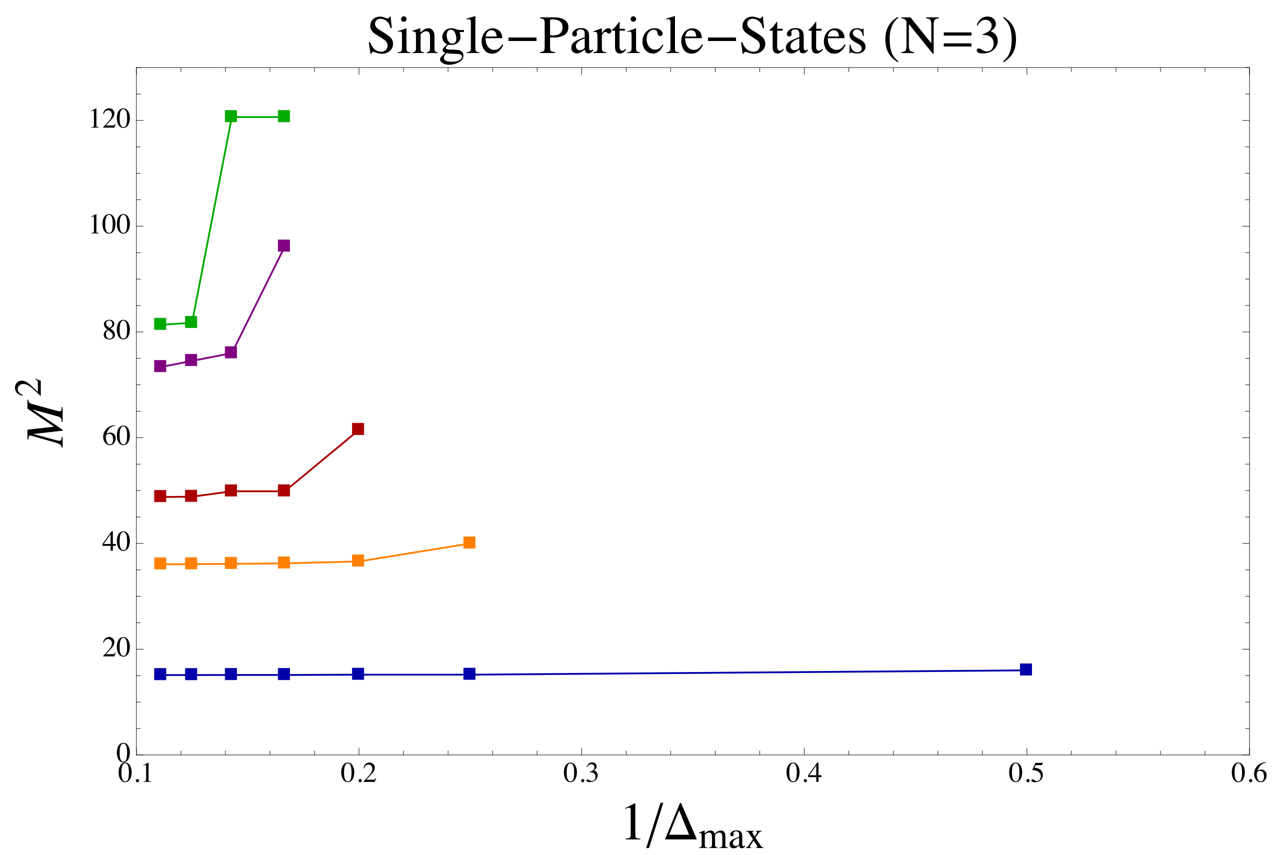
$$\mathcal{O}^{(2)} \sim (\partial \psi^\dagger)\psi - \psi^\dagger \partial \psi,$$

$$\mathcal{O}^{(3)} \sim (\psi^\dagger \psi)^2,$$

$$\mathcal{O}^{(4)} \sim (\partial \psi^\dagger)\psi\psi^\dagger\psi - \psi^\dagger(\partial\psi)\psi^\dagger\psi + \psi^\dagger\psi(\partial\psi^\dagger)\psi - \psi^\dagger\psi\psi^\dagger(\partial\psi).$$

$$M^2 = \frac{g^2 N}{2\pi} \begin{pmatrix} 10.7 & 7.54 & 2.72 \\ 7.54 & 5.33 & 1.92 \\ 2.72 & 1.92 & 45.6 \end{pmatrix}$$ (N=3)

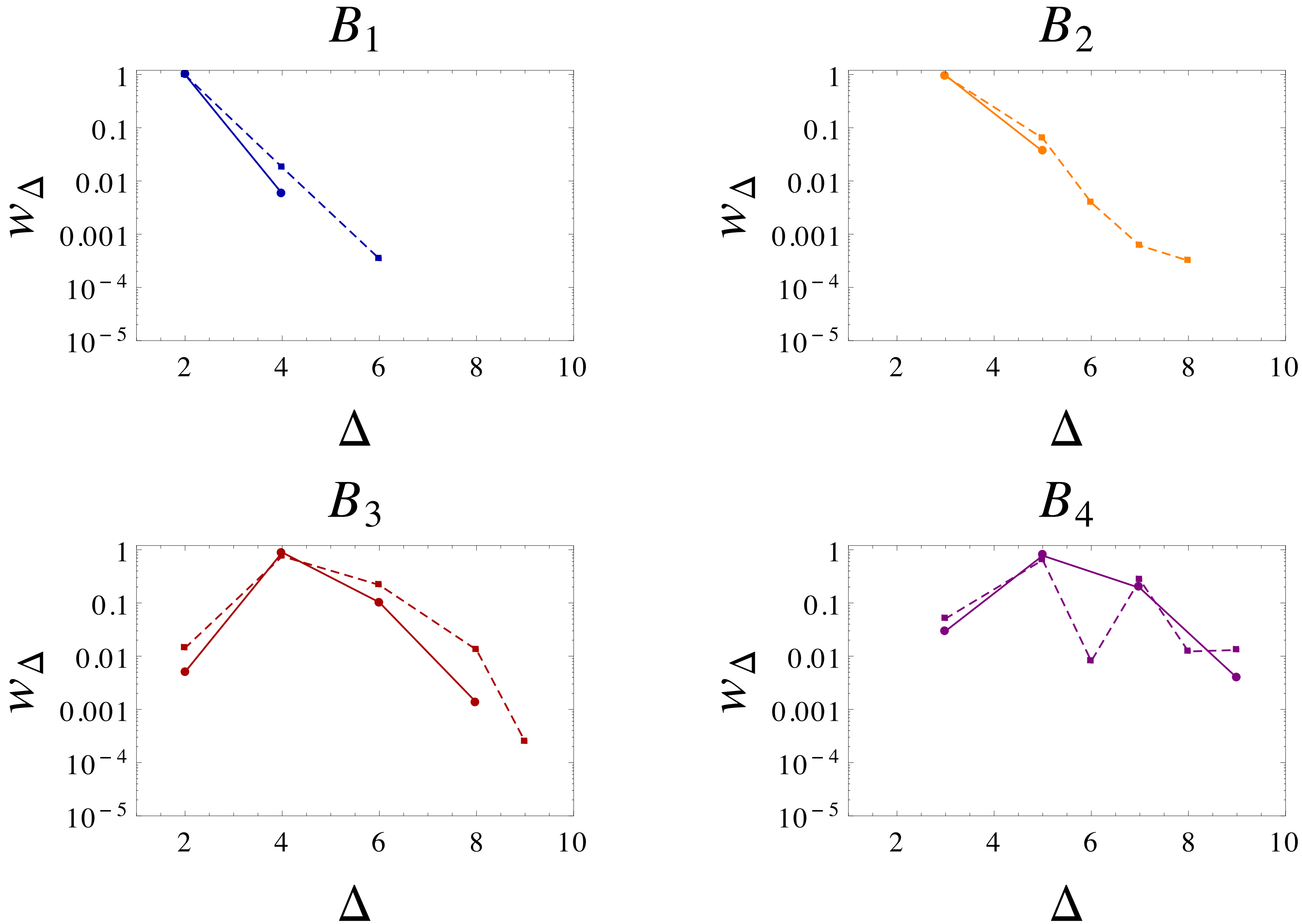


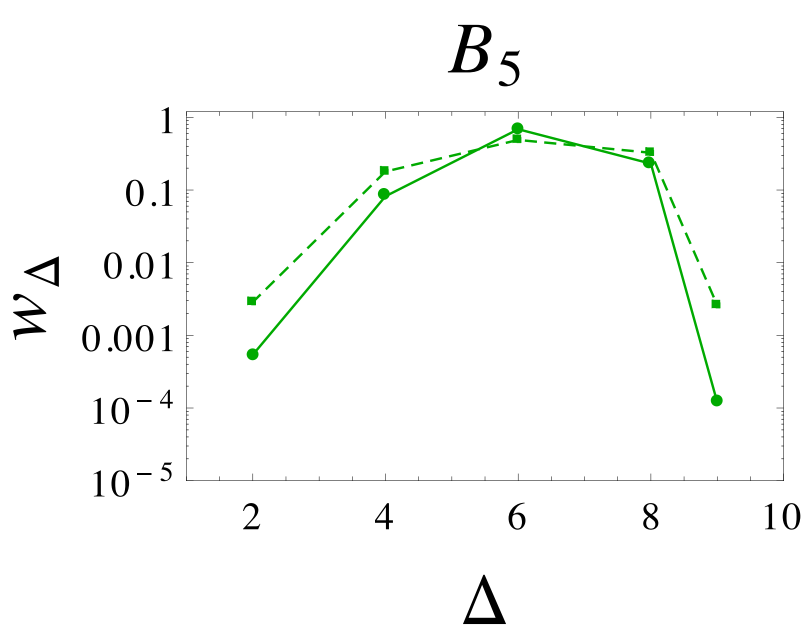


Ex N=3: $$|B_1\rangle = 0.81\left(\sqrt{3}(\partial\psi^\dagger\psi - \psi^\dagger\partial\psi)\right)|\Omega\rangle - 0.57\left(\frac{3}{\sqrt{2}}(\psi^\dagger\psi)^2\right)|\Omega\rangle$$

(up to ~1% corrections)

$$\mathrm{Weight}(\Delta) \equiv \sum_{\Delta_i = \Delta} \langle \Omega | \mathcal{O}_i | \psi \rangle^2$$





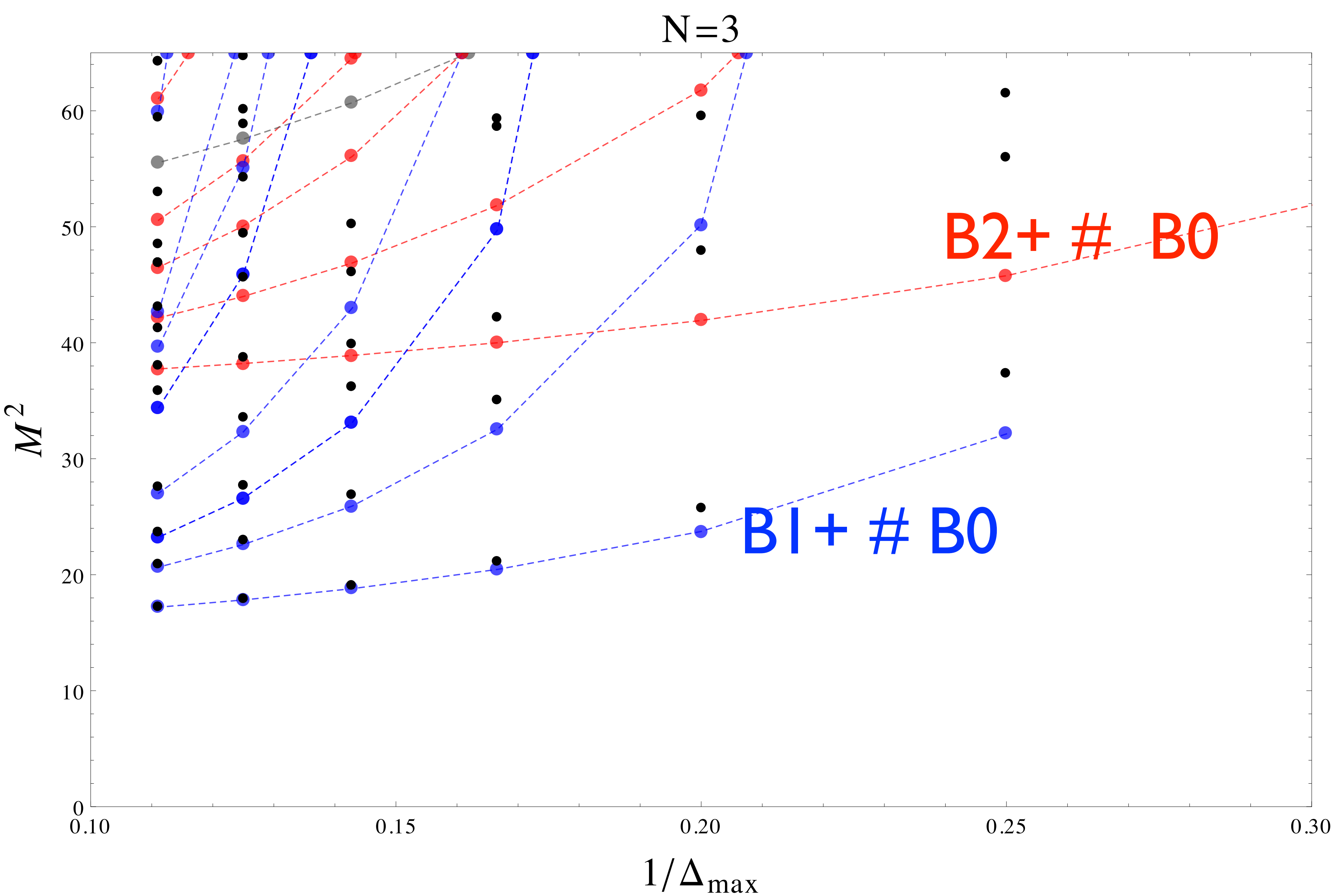
Solid-line: N=1000

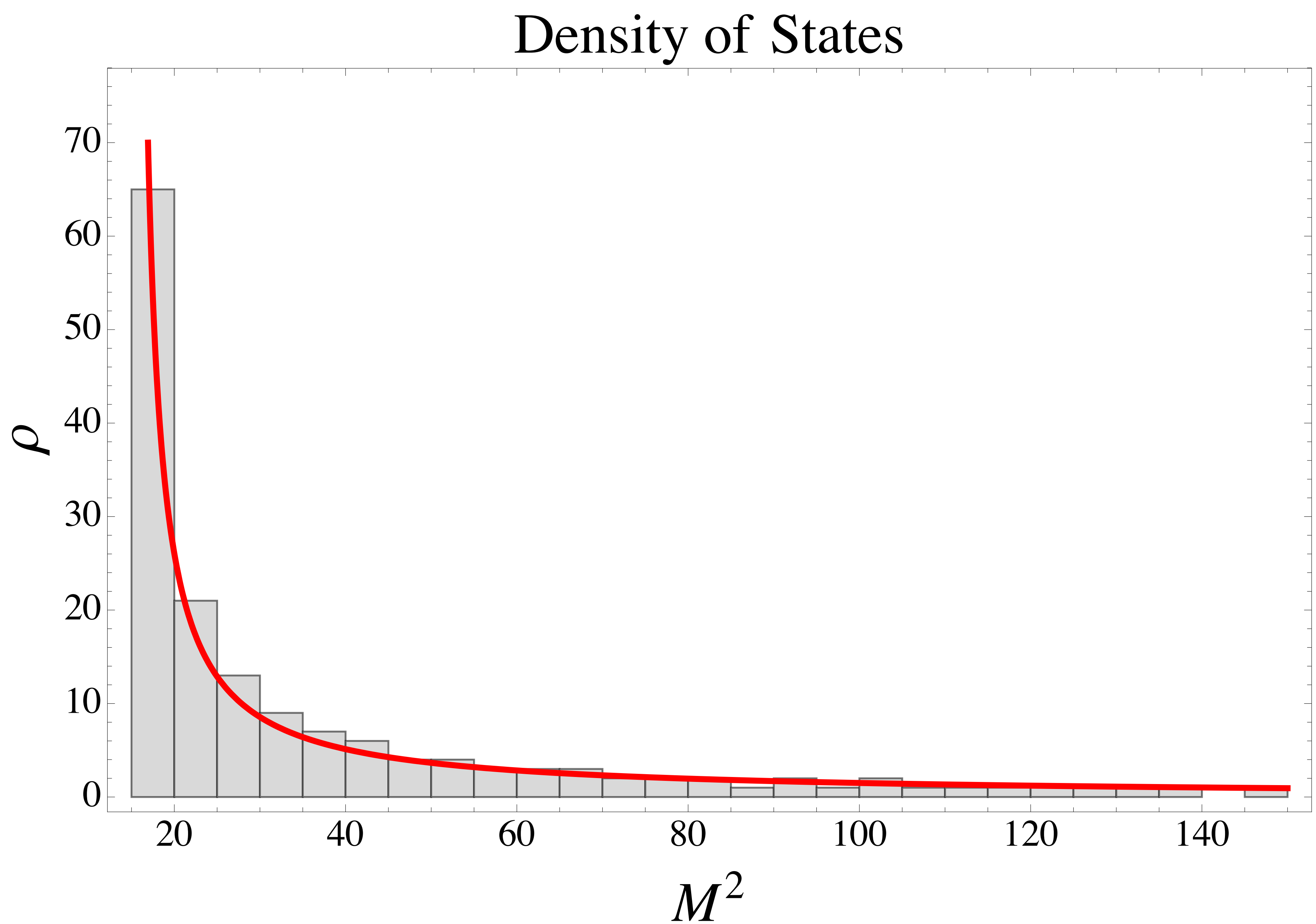
Dashed-line: N=3

Multi-particle states for 2D QCD at finite-N

Has a massless particle → many multi-part states:

$$B_1 + n_1 B_0,\ B_2 + n_2 B_0,\ B_3 + n_3 B_0, \cdots$$





$$\rho(M^2) = \frac{Z}{M^2 - M_{B_1}^2}. \qquad \Delta_{max} = 200$$

Few comments:

1. Using a conformal basis offers a way to non-perturbatively define the 2D gauge theory.

2. It is a discretization which naturally uses CFT discreteness without the need to introduce additional "external" deformations of the theory (like on the lattice).

3. It is effective for the low-energy spectrum, (Light 2D QCD states understood analytically)

4. How to estimate rapidity of convergence?

Free 3D scalar

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 \; + \frac{1}{2}m^2\phi^2$$

Issue: $\langle \mathcal{O}_i(\vec{p})|\mathcal{O}_j(\vec{p'})\rangle = \delta_{ij}\delta^2(p-p')$ is no longer finite!

$$|\mathcal{O}\rangle = \int \prod_i \frac{d^2 p_i}{p_{i-}} \; f_{\mathcal{O}}(\vec{p}_1, ..., \vec{p}_n)|\vec{p}_1, ..\vec{p}_n\rangle$$

(Integration over $p_\perp$ no longer compact)

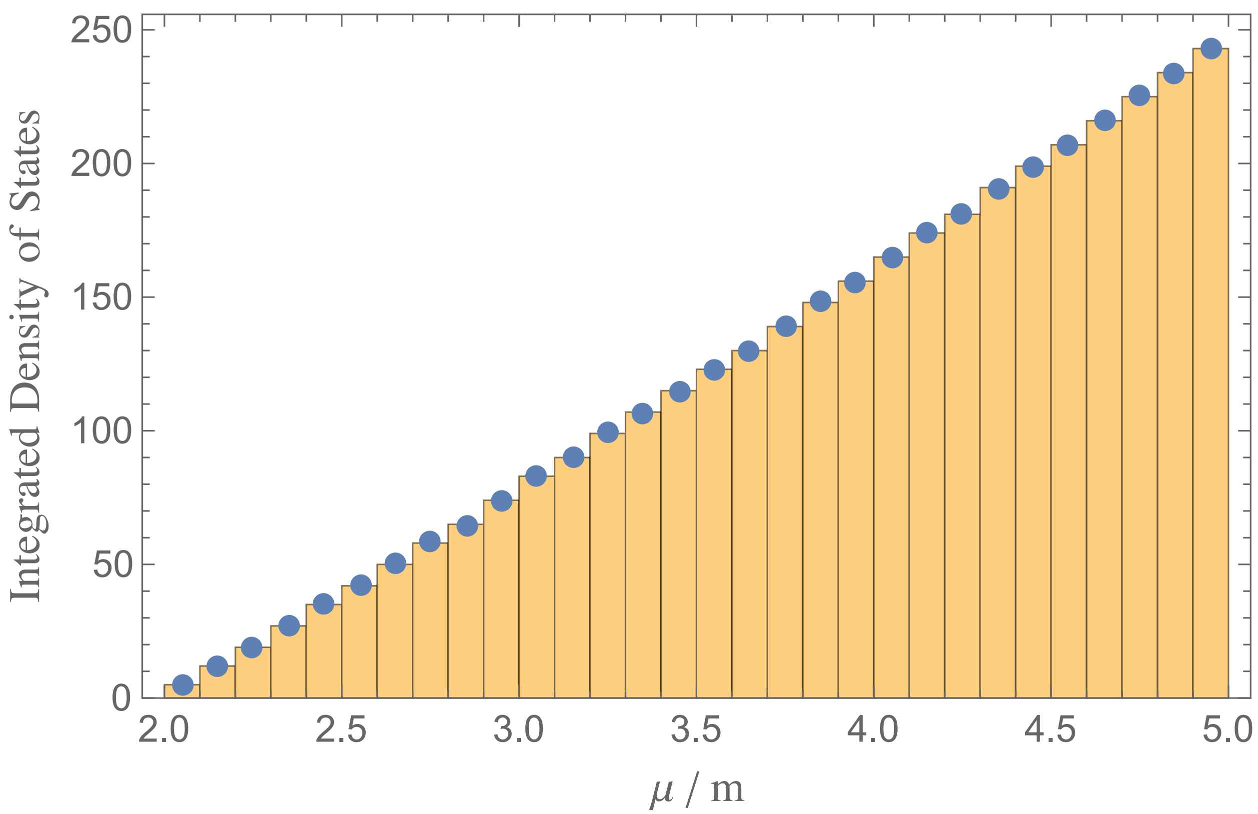
Cure: CFT respecting regulator: $P_+P_- < \Lambda^2$

Ex: n-particle sector $P_+ = \frac{p_{1\perp}^2}{2p_{1-}} + \cdots + \frac{p_{n\perp}^2}{2(P_- - \sum_i p_{i-})}$

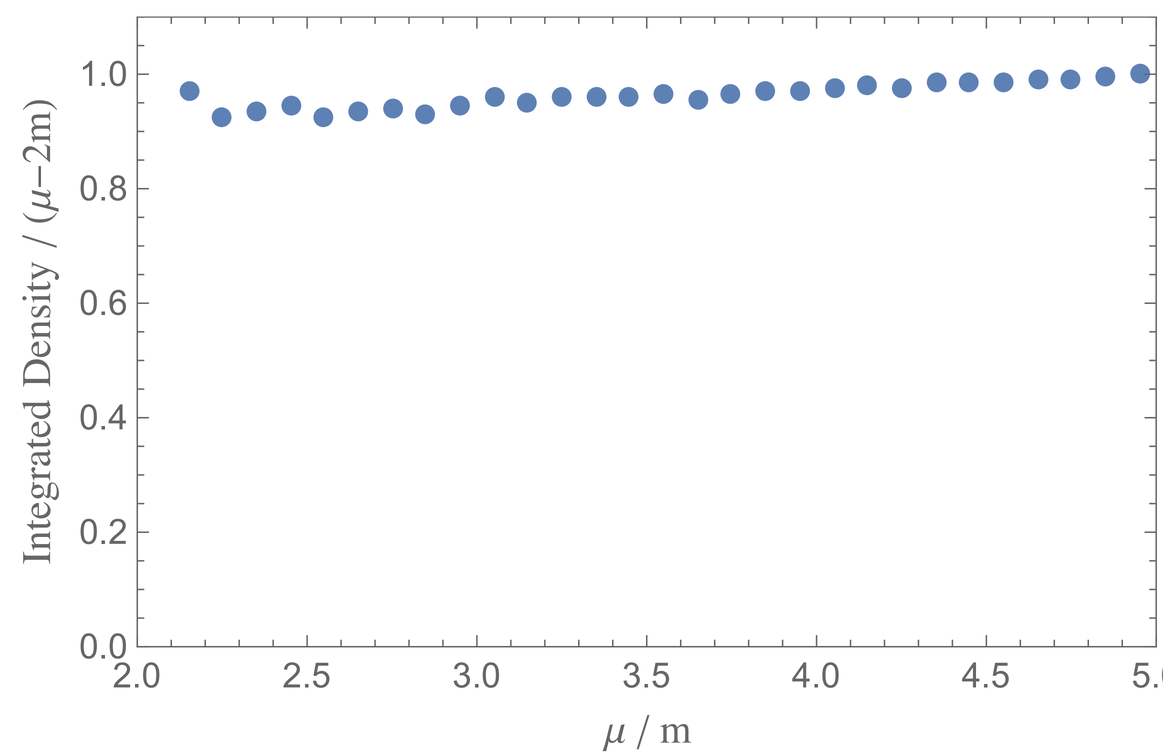
For the 2-particle state, the density is

$$\rho(\mu^2) \sim \frac{1}{\mu} \quad \Rightarrow \quad \int_{4m^2}^{\mu^2} d\mu'^2 \; \rho(\mu'^2) \sim (\mu - 2m)$$

$$m = 1, \;\; \Lambda = 10 :$$

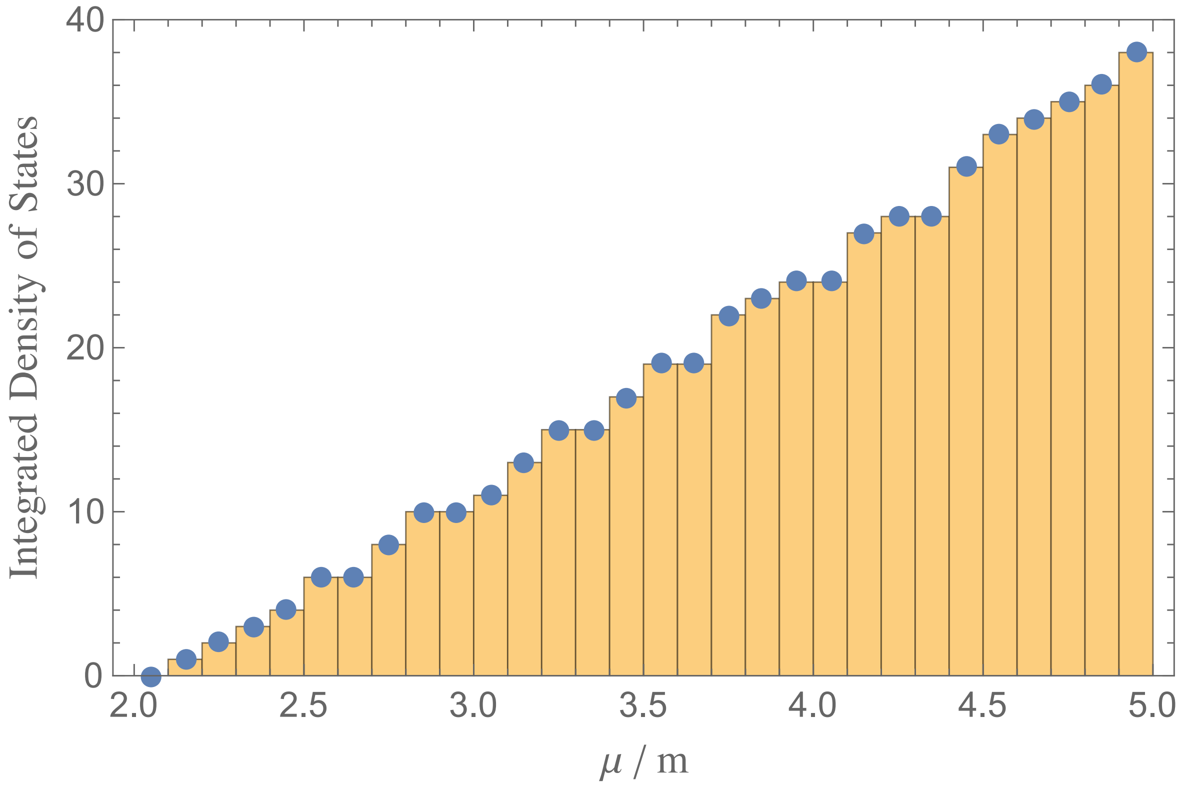


$$\Delta_{max} = 50$$

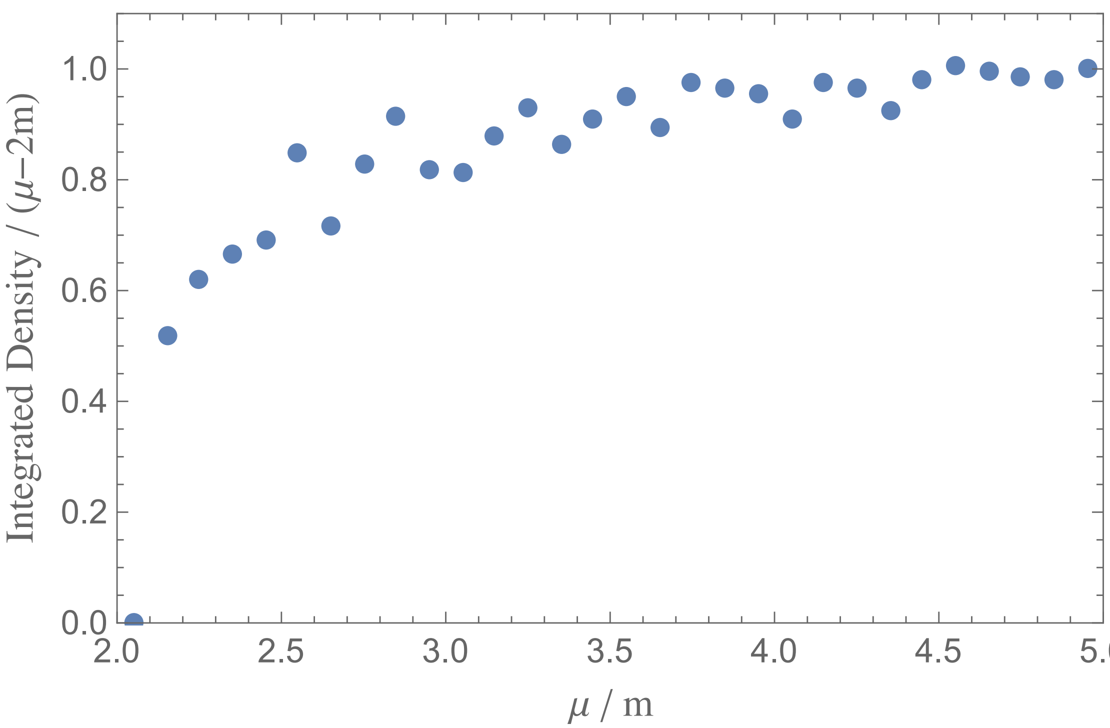


power fit: $(\mu - 2m)^{1.05}$

$$m = 1,\ \Lambda = 10 :$$

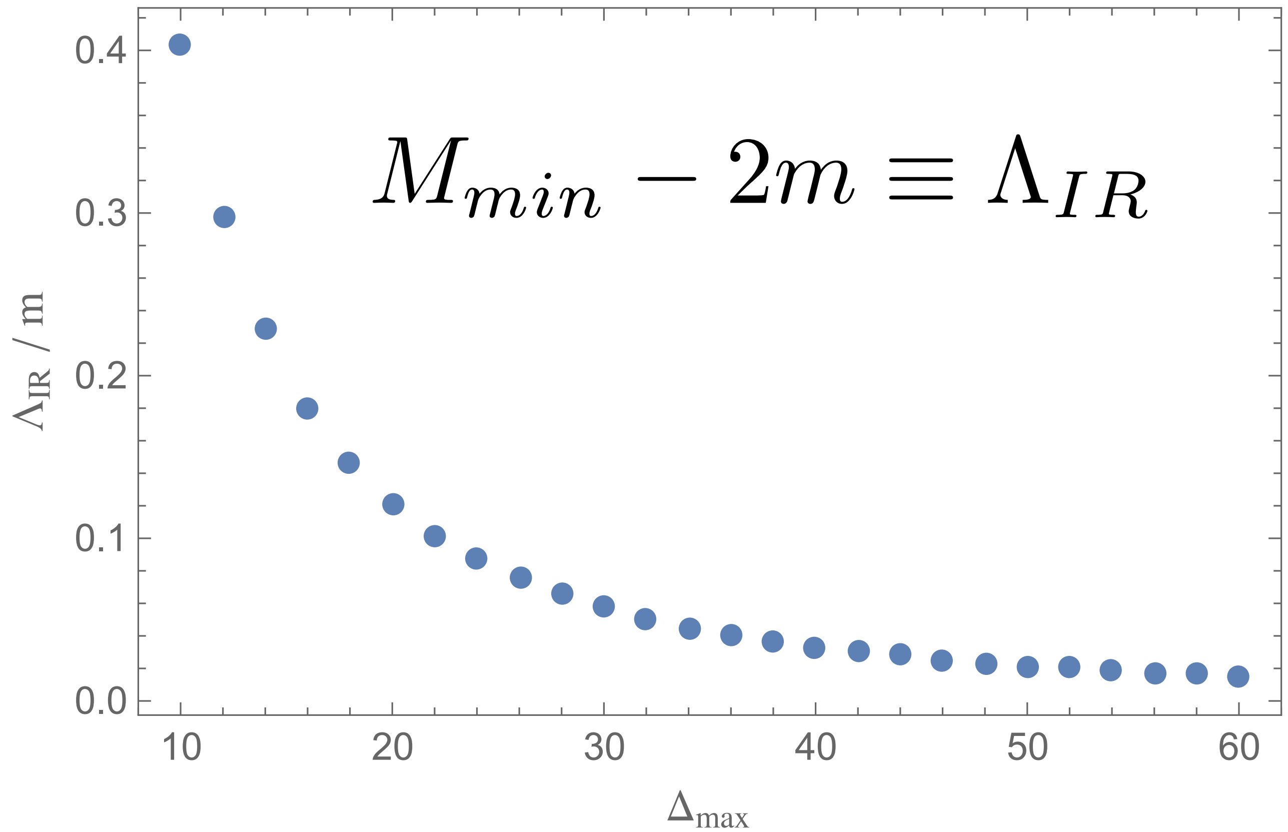


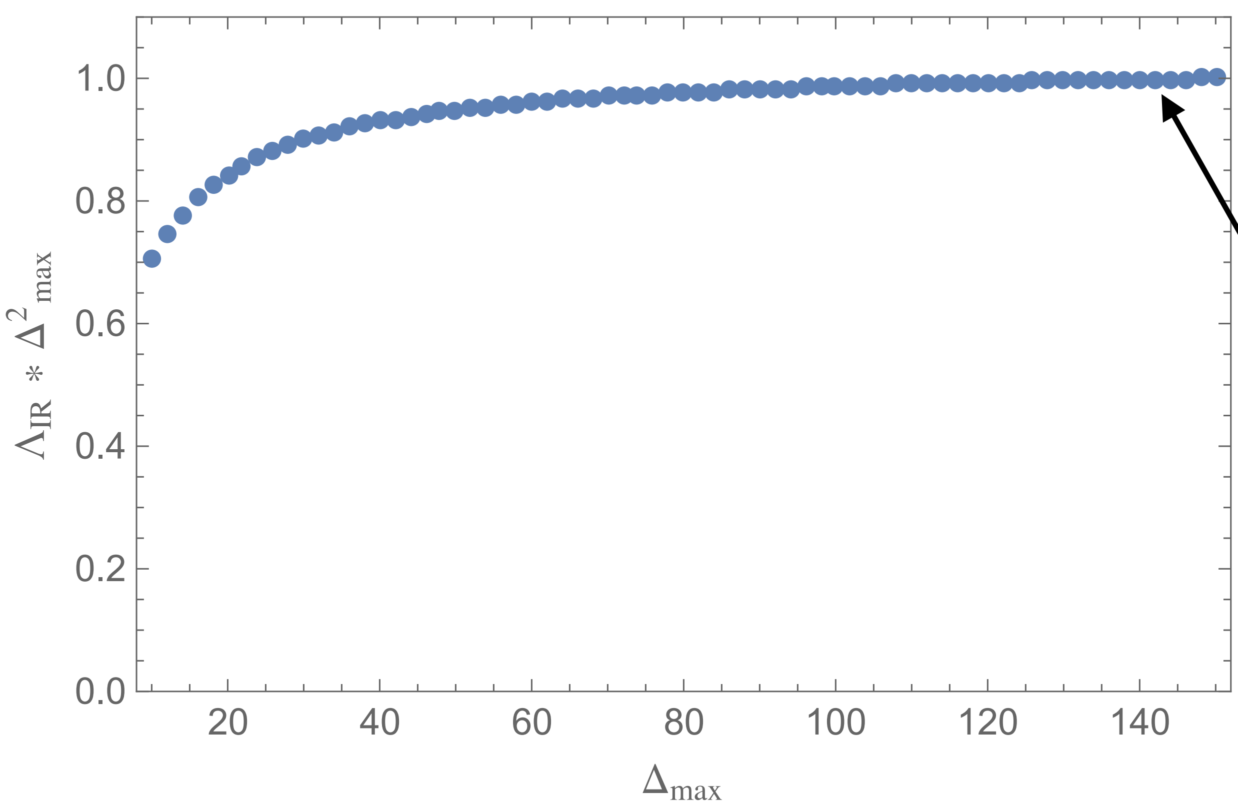
$$\Delta_{max} = 20$$



power fit: $(\mu - 2m)^{1.14}$

So what is the size of the discretization error?





Parametrically:

$$\Lambda_{IR} \sim \frac{1}{\Delta^2}$$

Extending the construction to non-Lagrangian theories

Problem - on the LC there are constraints that need to be implemented.

Ex - massive fermion in 2d: $\partial_- \chi = \frac{m}{2}\psi$

How to incorporate constraints when there's no EOM?

Use OPE of the relevant operator with other ops.

$$i \oint_{x^+=0} dx^+ dx_1 \, \lambda \mathcal{O}_R(x^+, x_1)\, \mathcal{O}_i(0, x_2) = \sum_j c_{Rij} \int dx_1 f_j(x_{12}) \mathcal{O}_j(0, x_2)$$

Gives $\mathcal{O}_i$ in terms of $\mathcal{O}_j$

Ex - massive fermion: $\lambda \mathcal{O}_R \sim m(\chi\psi)$

Implementing the constraint OPE gives:

$$\chi(0, x_2) = \frac{m}{4} \int dx_1 \ sgn(x_2 - x_1)\psi(0, x_1)$$

(consistent with the EOM)

OPE of relevant op with itself determines hamiltonian.

2D QCD examples can formulated this way.

In practice, for non-integer dimensions, the procedure requires a regulator.

Interesting open problem!

Conclusions/Confusions

1. There's new approach to solving/quantizing a QFT using a conformal basis on the LC.
2. It is based on the decoupling of high scaling dimension ops from low-E spectrum: "Effective Conformal Dominance".
3. Evidence for exponential decoupling in gapped strongly coupled 2D systems at small N and with a discrete spectrum of bound-states.
4. Can be formulated in 3D. For a free scalar, when spectrum was continuous, we saw power-law decoupling.
   (Currently working on extension to $V = m^2\phi^2 + \lambda\phi^4$)
5. Many open questions:

   How can we estimate the rate of decoupling? Is it related to the behavior of the density of states near the gap?

   How to deal with Non-Lagrangian theories?